Section 4.1
Angles and Radian Measure

Ever Feel Like You're Just Going in Circles?

You're riding on a Ferris wheel and wonder how fast you are traveling. Before you got on the ride, the operator told you that the wheel completes two full revolutions every minute and that your seat is 25 feet from the center of the wheel. You just rode on the merry-go-round, which made 2.5 complete revolutions per minute. Your wooden horse was 20 feet from the center, but your friend, riding beside you was only 15 feet from the center. Were you and your friend traveling at the same rate?

In this section, we study both angular speed and linear speed and solve problems similar to those just stated.

Objective #1: Recognize and use the vocabulary of angles.

✔ *Solved Problem #1*

1a. True or false: When an angle is in standard position, its initial side is along the positive y-axis.

False; When an angle is in standard position, its initial side is along the positive x-axis.

1b. Fill in the blank to make a true statement: If the terminal side of an angle in standard position lies on the x-axis or the y-axis, the angle is called a/an ___________ angle.

Such an angle is called a quadrantal angle.

✎ *Pencil Problem #1* ✐

1a. True or false: When an angle is in standard position, its vertex lies in quadrant I.

1b. Fill in the blank to make a true statement: A negative angle is generated by a __________ rotation.

Objective #2: Use degree measure.

✔ *Solved Problem #2*

2. Fill in the blank to make a true statement: An angle that is formed by $\frac{1}{2}$ of a complete rotation measures _________ degrees and is called a/an _________ angle.

Such an angle measures 180 degrees and is called a straight angle.

✎ *Pencil Problem #2* ✐

2. Fill in the blank to make a true statement: An angle that is formed by $\frac{1}{4}$ of a complete rotation measures _________ degrees and is called a/an _________ angle.

Objective #3: Use radian measure.

✔ Solved Problem #3

3. A central angle, θ, in a circle of radius 12 feet intercepts an arc of length 42 feet. What is the radian measure of θ?

The radian measure of the central angle, θ, is the length of the intercepted arc, s, divided by the radius of the circle, r: $\theta = \frac{s}{r}$. In this case, $s = 42$ feet and $r = 12$ feet.

$$\theta = \frac{s}{r} = \frac{42 \ \cancel{\text{feet}}}{12 \ \cancel{\text{feet}}} = 3.5$$

The radian measure of θ is 3.5.

✎ Pencil Problem #3 ✐

3. A central angle, θ, in a circle of radius 10 inches intercepts an arc of length 40 inches. What is the radian measure of θ?

Objective #4: Convert between degrees and radians.

✔ Solved Problem #4

4a. Convert 60° to radians.
To convert from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$. Then simplify.

$$60\cancel{°} \cdot \frac{\pi \text{ radians}}{180\cancel{°}} = \frac{60\pi \text{ radians}}{180} = \frac{\pi}{3} \text{ radians}$$

✎ Pencil Problem #4 ✐

4a. Convert 135° to radians. Express your answer as a multiple of π.

4b. Convert −300° to radians.

$$-300\cancel{°} \cdot \frac{\pi \text{ radians}}{180\cancel{°}} = -\frac{300\pi \text{ radians}}{180} = -\frac{5\pi}{3} \text{ radians}$$

4b. Convert −225° to radians. Express your answer as a multiple of π.

4c. Convert $\frac{\pi}{4}$ radians to degrees.

To convert from radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$. Then simplify.

$$\frac{\cancel{\pi}}{4} \cancel{\text{radians}} \cdot \frac{180°}{\cancel{\pi} \cancel{\text{radians}}} = \frac{180°}{4} = 45°$$

4c. Convert $\frac{\pi}{2}$ radians to degrees.

4d. Convert 6 radians to degrees.

$$6 \cancel{\text{radians}} \cdot \frac{180°}{\pi \cancel{\text{radians}}} = \frac{1080°}{\pi} \approx 343.8°$$

4d. Convert 2 radians to degrees. Round to two decimal places.

Objective #5: Draw angles in standard position.

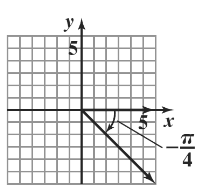
✔ *Solved Problem #5*

5a. Draw the angle $\theta = -\frac{\pi}{4}$ in standard position.

Since the angle is negative, it is obtained by a clockwise rotation. Express the angle as a fractional part of 2π.

$$\left|-\frac{\pi}{4}\right| = \frac{\pi}{4} = \frac{1}{8} \cdot 2\pi$$

The angle $\theta = -\frac{\pi}{4}$ is $\frac{1}{8}$ of a full rotation in the clockwise direction.



✎ *Pencil Problem #5* ✎

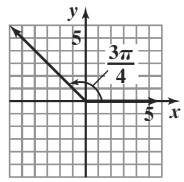
5a. Draw the angle $\theta = -\frac{5\pi}{4}$ in standard position.

5b. Draw the angle $\alpha = \frac{3\pi}{4}$ in standard position.

Since the angle is positive, it is obtained by a counterclockwise rotation. Express the angle as a fractional part of 2π.

$\frac{3\pi}{4} = \frac{3}{8} \cdot 2\pi$

The angle $\alpha = \frac{3\pi}{4}$ is $\frac{3}{8}$ of a full rotation in the counterclockwise direction.

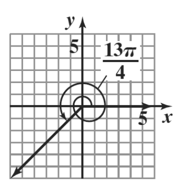


5c. Draw the angle $\gamma = \frac{13\pi}{4}$ in standard position.

Since the angle is positive, it is obtained by a counterclockwise rotation. Express the angle as a fractional part of 2π.

$\frac{13\pi}{4} = \frac{13}{8} \cdot 2\pi$

The angle $\gamma = \frac{13\pi}{4}$ is $\frac{13}{8}$ or $1\frac{5}{8}$ full rotation in the counterclockwise direction. Complete one full rotation and then $\frac{5}{8}$ of a full rotation.



5b. Draw the angle $\alpha = \frac{7\pi}{6}$ in standard position.

5c. Draw the angle $\gamma = \frac{16\pi}{3}$ in standard position.

Objective #6: Find coterminal angles.

✔ *Solved Problem #6*

6a. Find a positive angle less than 360° that is coterminal with a 400° angle.

Since 400° is greater than 360°, we subtract 360°.

$400° - 360° = 40°$

A 40° angle is positive, less than 360°, and coterminal with a 400° angle.

6b. Find a positive angle less than 2π that is coterminal with a $-\frac{\pi}{15}$ angle.

Since $-\frac{\pi}{15}$ is negative, we add 2π.

$$-\frac{\pi}{15}+2\pi=-\frac{\pi}{15}+\frac{30\pi}{15}=\frac{29\pi}{15}$$

A $\frac{29\pi}{15}$ angle is positive, less than 2π, and coterminal with a $-\frac{\pi}{15}$ angle.

✎ *Pencil Problem #6* ✎

6a. Find a positive angle less than 360° that is coterminal with a 395° angle.

6b. Find a positive angle less than 2π that is coterminal with a $-\frac{\pi}{50}$ angle.

6c. Find a positive angle less than 2π that is coterminal with a $\frac{17\pi}{3}$ angle.

Since $\frac{17\pi}{3}$ is greater than 4π, we subtract two multiples of 2π.

$$\frac{17\pi}{3} - 2 \cdot 2\pi = \frac{17\pi}{3} - 4\pi = \frac{17\pi}{3} - \frac{12\pi}{3} = \frac{5\pi}{3}$$

A $\frac{5\pi}{3}$ angle is positive, less than 2π, and coterminal with a $\frac{17\pi}{3}$ angle.

6c. Find a positive angle less than 2π that is coterminal with a $-\frac{31\pi}{7}$ angle.

Objective #7: Find the length of a circular arc.

✔ *Solved Problem #7*

7. A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of 45°. Express arc length in terms of π. Then round your answer to two decimal places.

We begin by converting 45° to radians.

$$45\cancel{°} \cdot \frac{\pi \text{ radians}}{180\cancel{°}} = \frac{45\pi \text{ radians}}{180} = \frac{\pi}{4} \text{ radians}$$

Now we use the arc length formula $s = r\theta$ with the radius $r = 6$ inches and the angle $\theta = \frac{\pi}{4}$ radians.

$$s = r\theta = (6 \text{ in.})\left(\frac{\pi}{4}\right) = \frac{6\pi}{4} \text{ in.} = \frac{3\pi}{2} \text{ in.} \approx 4.71 \text{ in.}$$

✎ *Pencil Problem #7* ✐

7. A circle has a radius of 8 feet. Find the length of the arc intercepted by a central angle of 225°. Express arc length in terms of π. Then round your answer to two decimal places.

Objective #8: Use linear and angular speed to describe motion on a circular path.

✔ Solved Problem #8

8. A 45-rpm record has an angular speed of 45 revolutions per minute. Find the linear speed, in inches per minute, at the point where the needle is 1.5 inches from the record's center.

We are given the angular speed in revolutions per minute: $\omega = 45$ revolutions per minute. We must express ω in radians per minute.

$$\omega = \frac{45 \cancel{\text{ revolutions}}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \cancel{\text{ revolution}}}$$
$$= \frac{90\pi \text{ radians}}{1 \text{ minute}} \text{ or } \frac{90\pi}{1 \text{ minute}}$$

Now we use the formula $v = r\omega$.

$$v = r\omega = 1.5 \text{ in.} \cdot \frac{90\pi}{1 \text{ min}} = \frac{135\pi \text{ in.}}{\text{min}} \approx 424 \text{ in./min}$$

✎ Pencil Problem #8 ✎

8. A Ferris wheel has a radius of 25 feet. The wheel is rotating at two revolutions per minute. Find the linear speed, in feet per minute, of a seat on this Ferris wheel.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

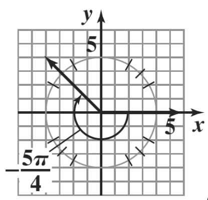
1a. False **1b.** clockwise

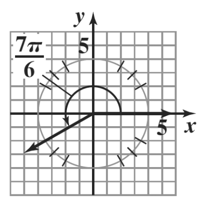
2. 90; right

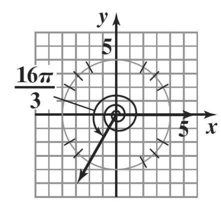
3. 4 radians *(4.1 #7)*

4a. $\frac{3\pi}{4}$ radians *(4.1 #15)* **4b.** $-\frac{5\pi}{4}$ radians *(4.1 #19)* **4c.** 90° *(4.1 #21)*

4d. 114.59° *(4.1 #35)*

5a. 

(4.1 #47) **5b.** 

(4.1 #41) **5c.** 

(4.1 #49)

6a. 35° *(4.1 #57)* **6b.** $\frac{99\pi}{50}$ *(4.1 #67)* **6c.** $\frac{11\pi}{7}$ *(4.1 #69)*

7. 10π ft ≈ 31.42 ft *(4.1 #73)*

8. 100π ft/min ≈ 314 ft/min *(4.1 #98)*

Section 4.2
Trigonometric Functions: The Unit Circle

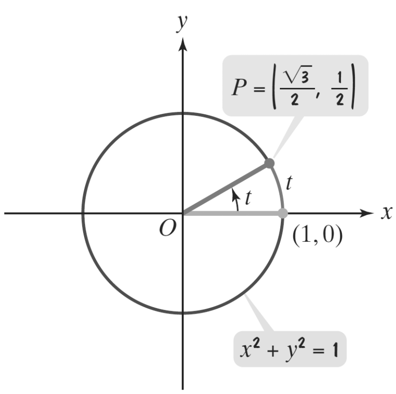
Could you repeat that?

There are many repetitive patterns in nature. Tides cycle through a pattern of low and high tides in a very predictable manner. The number of hours of daylight on a given day varies throughout the year, but the pattern throughout the year repeats itself year after year.
In this section, we define the *trigonometric functions* using movement around a unit circle. In doing so, we can see that the values of the trigonometric functions repeat themselves each time we make a complete trip around the circle. The repetitive properties of the trigonometric functions make them useful for modeling cyclic phenomena.

Objective #1: Use a unit circle to define trigonometric functions of real numbers.

✔ *Solved Problem #1*

1. Use the figure to find the values of the trigonometric functions at t.



The point on the circle has coordinates $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

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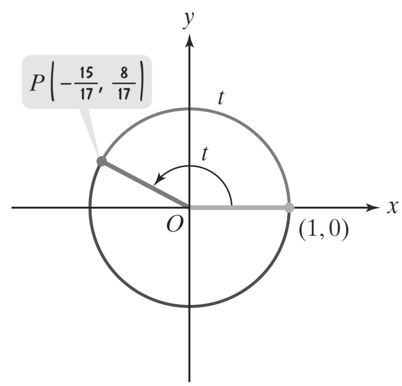
✎ *Pencil Problem #1* ✎

1. Use the figure to find the values of the trigonometric functions at t.



We use $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$ in the definitions.

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = \frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\sec t = \frac{1}{x} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}}$$

$$\cot t = \frac{x}{y} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Objective #2: Recognize the domain and range of sine and cosine functions.

✔ *Solved Problem #2*

2. True or false: The range of the cosine function is all real numbers, $(-\infty, \infty)$, so there is a real number t in the domain of the cosine function for which $\cos t = 10$.

False; the range of cosine is $[-1, 1]$ and 10 is not in this interval, so there is no real number t in the domain of the cosine function for which $\cos t = 10$.

✎ *Pencil Problem #2* ✐

2. True or false: There is a real number t in the domain of the sine function for which $\sin t = -\frac{\sqrt{10}}{2}$.

Objective #3: Find exact values of the trigonometric functions at $\frac{\pi}{4}$.

✔ *Solved Problem #3*

3. Find $\sin\frac{\pi}{4}$ and $\sec\frac{\pi}{4}$.

The point P on the unit circle that corresponds to $t = \frac{\pi}{4}$ has coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. We use $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ in the definitions of sine and secant.

$$\sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2}$$

$$\sec\frac{\pi}{4} = \frac{1}{x} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

✎ *Pencil Problem #3* ✎

3. Find $\csc\frac{\pi}{4}$ and $\tan\frac{\pi}{4}$.

Objective #4: Use even and odd trigonometric functions.

✔ *Solved Problem #4*

4a. Given that $\sec\frac{\pi}{4} = \sqrt{2}$, find $\sec\left(-\frac{\pi}{4}\right)$.

Since secant is an even function, $\sec(-t) = \sec t$.

$$\sec\left(-\frac{\pi}{4}\right) = \sec\frac{\pi}{4} = \sqrt{2}$$

4b. Given that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, find $\sin\left(-\frac{\pi}{4}\right)$.

Since sine is an odd function, $\sin(-t) = -\sin t$.

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

✎ *Pencil Problem #4* ✎

4a. Given that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, find $\cos\left(-\frac{\pi}{6}\right)$.

4b. Given that $\tan\frac{5\pi}{3} = -\sqrt{3}$, find $\tan\left(-\frac{5\pi}{3}\right)$.

Objective #5: Recognize and use fundamental identities.

✔ *Solved Problem #5*

5a. Given $\sin t = \frac{2}{3}$ and $\cos t = \frac{\sqrt{5}}{3}$, find the value of each of the four remaining trigonometric functions.

Find $\tan t$ using a quotient identity.

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}} = \frac{2}{3} \cdot \frac{3}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Use reciprocal identities to find the remaining three function values.

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{\sqrt{5}}{3}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}$$

5b. Given that $\sin t = \frac{1}{2}$ and $0 \le t < \frac{\pi}{2}$, find the value of $\cos t$ using a trigonometric identity.

Use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$.

Because $0 \le t < \frac{\pi}{2}$, $\cos t$ is positive.

$$\left(\frac{1}{2}\right)^2 + \cos^2 t = 1$$

$$\frac{1}{4} + \cos^2 t = 1$$

$$\cos^2 t = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\cos t = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

✎ *Pencil Problem #5* ✎

5a. Given $\sin t = \frac{1}{3}$ and $\cos t = \frac{2\sqrt{2}}{3}$, find the value of each of the four remaining trigonometric functions.

5b. Given that $\sin t = \frac{6}{7}$ and $0 \le t < \frac{\pi}{2}$, find the value of $\cos t$ using a trigonometric identity.

Objective #6: Use periodic properties.

✔ *Solved Problem #6*

6a. Find the value of $\cot\frac{5\pi}{4}$.

The period of cotangent is π: $\cot(t+\pi)=\cot t$.

$$\cot\frac{5\pi}{4}=\cot\left(\frac{\pi}{4}+\pi\right)=\cot\frac{\pi}{4}=1$$

6b. Find the value of $\cos\left(-\frac{9\pi}{4}\right)$.

The cosine function is even: $\cos(-t)=\cos t$.

The period of cosine is 2π: $\cos(t+2\pi)=\cos t$.

$$\cos\left(-\frac{9\pi}{4}\right)=\cos\frac{9\pi}{4}=\cos\left(\frac{\pi}{4}+2\pi\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

✎ *Pencil Problem #6* ✎

6a. Find the value of $\tan\frac{5\pi}{4}$.

6b. Find the value of $\sin\left(-\frac{9\pi}{4}\right)$.

Objective #7: Evaluate trigonometric functions with a calculator.

✔ *Solved Problem #7*

7a. Use a calculator to find the value of $\sin\frac{\pi}{4}$ to four decimal places.

Use radian mode.

On a scientific calculator, enter $\pi \div 4$, and then press = and the SIN key.

On a graphing calculator, press the SIN key, and then enter $\pi \div 4$, and press ENTER.

The display, rounded to four places, should be 0.7071.

✎ *Pencil Problem #7* ✎

7a. Use a calculator to find the value of $\cos\frac{\pi}{10}$ to four decimal places.

7b. Use a calculator to find the value of $\csc 1.5$ to four decimal places.

Use radian mode.

On a scientific calculator, enter 1.5, and then press the SIN key followed by the reciprocal key labeled $1/x$.

On a graphing calculator, open a set of parentheses, press the SIN key, and then enter 1.5. Close the parentheses, press the reciprocal key labeled x^{-1}, and press ENTER. The display, rounded to four places, should be 1.0025.

7b. Use a calculator to find the value of sec 1 to four decimal places.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\sin t = \frac{8}{17}$, $\cos t = -\frac{15}{17}$, $\tan t = -\frac{8}{15}$, $\csc t = \frac{17}{8}$, $\sec t = -\frac{17}{15}$, $\cot t = -\frac{15}{8}$ *(4.2 #1)*

2. false

3. $\csc\frac{\pi}{4} = \sqrt{2}$; $\tan\frac{\pi}{4} = 1$ *(4.2 Check Point #3)*

4a. $\frac{\sqrt{3}}{2}$ *(4.2 #19b)* **4b.** $\sqrt{3}$ *(4.2 #23b)*

5a. $\tan t = \frac{\sqrt{2}}{4}$, $\csc t = 3$, $\sec t = \frac{3\sqrt{2}}{4}$, $\cot t = 2\sqrt{2}$ *(4.2 #27)* **5b.** $\frac{\sqrt{13}}{7}$ *(4.2 #29)*

6a. 1 *(4.2 #43)* **6b.** $-\frac{\sqrt{2}}{2}$ *(4.2 #41)*

7a. 0.9511 *(4.2 #67)* **7b.** 1.8508 *(4.2 #66)*

Section 4.3
Right Triangle Trigonometry

Measuring Up, Way Up

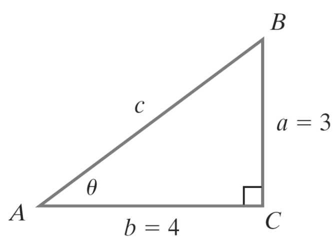
Did you ever wonder how you could measure the height of a building or a tree? How can you find the distance across a lake or some other body of water?

In this section, we show how to model such situations using a right triangle and then using relationships among the lengths its sides and the measures of its angles to find distances that are otherwise difficult to measure. These relationships lead to a second approach to the trigonometric functions.

Objective #1: Use right triangles to evaluate trigonometric functions.

✔ *Solved Problem #1*

1a. Find the value of each of the six trigonometric functions of θ in the figure.



We first need to find c, the length of the hypotenuse. We use the Pythagorean Theorem.

$$c^2 = a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25$$
$$c = \sqrt{25} = 5$$

We apply the right triangle definitions of the six trigonometric functions. Note that the side labeled $a = 3$ is opposite angle θ and the side labeled $b = 4$ is adjacent to angle θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

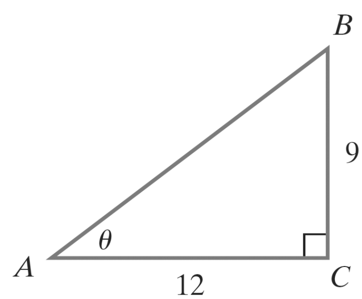
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}$$

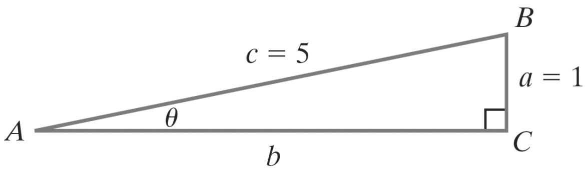
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

✎ *Pencil Problem #1* ✎

1a. Find the value of each of the six trigonometric functions of θ in the figure.



1b. Find the value of each of the six trigonometric functions of θ in the figure. Express each value in simplified form.



We first need to find b.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 5^2 \\ 1 + b^2 &= 25 \\ b^2 &= 24 \\ b &= \sqrt{24} = 2\sqrt{6} \end{aligned}$$

We apply the right triangle definitions of the six trigonometric functions.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{5}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{6}}{5}$$

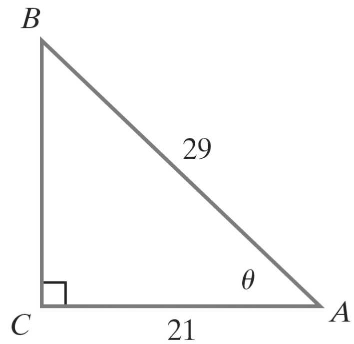
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{2\sqrt{6}} = \frac{1}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{12}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{1} = 5$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{2\sqrt{6}} = \frac{5}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{2\sqrt{6}}{1} = 2\sqrt{6}$$

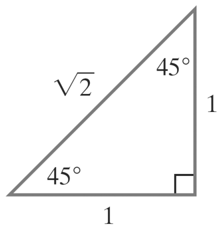
1b. Find the value of each of the six trigonometric functions of θ in the figure. Express each value in simplified form.



Objective #2: Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.

✔ Solved Problem #2

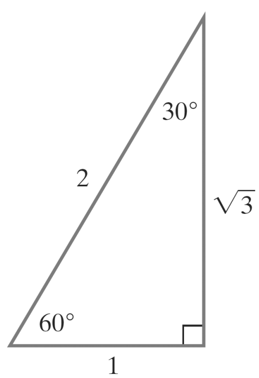
2a. Use the right triangle to find $\csc 45°$.



Use the definition of the cosecant function.

$$\csc 45° = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

2b. Use the right triangle to find $\tan 60°$.



Use the definition of the tangent function and the angle marked 60° in the triangle.

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

✎ Pencil Problem #2 ✎

2a. Use the right triangle in Solved Problem #2a to find $\sec 45°$.

2b. Use the right triangle in Solved Problem #2b to find $\cos 30°$.

Objective #3: Use equal cofunctions of complements.

✔ Solved Problem #3

3a. Find a cofunction with the same value as $\sin 46°$.

$$\sin 46° = \cos(90° - 46°) = \cos 44°$$

✎ Pencil Problem #3 ✎

3a. Find a cofunction with the same value as $\sin 7°$.

3b. Find a cofunction with the same value as $\cot\dfrac{\pi}{12}$.

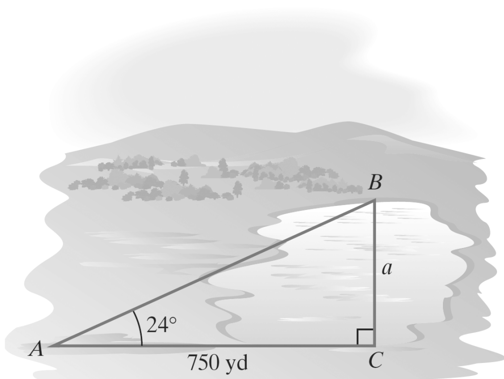
$$\cot\frac{\pi}{12} = \tan\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = \tan\left(\frac{6\pi}{12}-\frac{\pi}{12}\right) = \tan\left(\frac{5\pi}{12}\right)$$

3b. Find a cofunction with the same value as $\tan\dfrac{\pi}{9}$.

Objective #4: Use right triangle trigonometry to solve applied problems.

✔ *Solved Problem #4*

4. The distance across a lake, *a*, is unknown. To find this distance, a surveyor took the measurements shown in the figure. What is the distance across the lake?



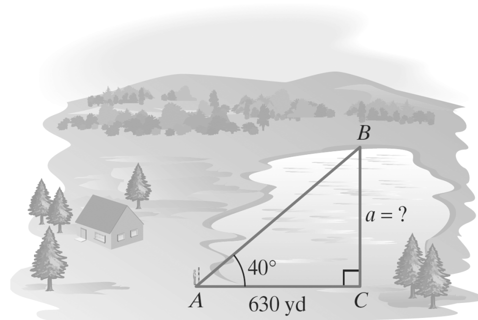
We know the measurements of one angle and the leg adjacent to the angle. We need to know the length of the side opposite the known angle. We use the tangent function.

$$\tan 24^\circ = \frac{a}{750}$$
$$a = 750\tan 24^\circ$$
$$a \approx 333.9$$

The distance across the lake is approximately 333.9 yards.

✎ *Pencil Problem #4* ✎

4. To find the distance across a lake, a surveyor took the measurements shown in the figure. Use the measurements to determine how far it is across the lake. Round to the nearest yard.



Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5},\ \tan\theta = \frac{3}{4},\ \csc\theta = \frac{5}{3},\ \sec\theta = \frac{5}{4},\ \cot\theta = \frac{4}{3}$ *(4.3 #1)*

1b. $\sin\theta = \frac{20}{29},\ \cos\theta = \frac{21}{29},\ \tan\theta = \frac{20}{21},\ \csc\theta = \frac{29}{20},\ \sec\theta = \frac{29}{21},\ \cot\theta = \frac{21}{20}$ *(4.3 #3)*

2a. $\sqrt{2}$ *(4.3 #11)* **2b.** $\frac{\sqrt{3}}{2}$ *(4.3 #9)*

3a. $\tan\theta = \frac{\sqrt{2}}{4},\ \csc\theta = 3,\ \sec\theta = \frac{3\sqrt{2}}{4},\ \cot\theta = 2\sqrt{2}$ *(4.3 #19)* **3b.** $\frac{\sqrt{13}}{7}$ *(4.3 #21)*

4a. $\cos 83^\circ$ *(4.3 #31)* **4b.** $\cot\frac{7\pi}{18}$ *(4.3 #35)*

5a. 0.6420 *(4.3 #41)* **5b.** 3.7321 *(4.3 #47)* **6.** 529 yd *(4.3 #73)*

Section 4.4
Trigonometric Functions of Any Angle

Covering All the Angles

We've already considered two ways of defining the six trigonometric functions. In the previous section, we defined the functions for acute angles of a right triangle. In this section, we extend the definitions of the trigonometric functions to include all angles.

Using this approach, even more patterns in the function values become apparent. These patterns allow us to evaluate the trigonometric functions more efficiently.

Objective #1: Use the definitions of the trigonometric functions of any angle.

✔ *Solved Problem #1*

1a. Let $P = (1, -3)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

We are given that $x = 1$ and $y = -3$. We need the value of r.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

Now we use the definitions of the trigonometric functions of any angle.

$$\sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{10}} = \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{1} = -3$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{1} = \sqrt{10}$$

$$\cot\theta = \frac{x}{y} = \frac{1}{-3} = -\frac{1}{3}$$

✎ *Pencil Problem #1* ✎

1a. Let $P = (-2, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

1b. Evaluate, if possible: $\csc 180°$.

The terminal side of $\theta = 180°$ is on the negative x-axis. We select the point $(-1, 0)$ on the terminal side of the angle, which is 1 unit from the origin, so $x = -1$, $y = 0$, and $r = 1$.

$$\csc\theta = \frac{r}{y} = \cancel{\frac{1}{0}}$$

$\csc 180°$ is undefined.

1b. Evaluate, if possible: $\tan\frac{3\pi}{2}$.

Objective #2: Use the signs of the trigonometric functions.

✔ *Solved Problem #2*

2a. If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which θ lies.

When $\sin\theta < 0$, θ lies in quadrant III or IV. When $\cos\theta < 0$, θ lies in quadrant II or III. When both conditions are met, θ must lie in quadrant III.

2b. Given that $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.

Because both the tangent and cosine are negative, θ lies in quadrant II, where x is negative and y is positive.

$$\tan\theta = -\frac{1}{3} = \frac{y}{x} = \frac{1}{-3}$$

So, $x = -3$ and $y = 1$. Find r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

Now use the definitions of the trigonometric functions of any angle.

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3}$$

✎ *Pencil Problem #2* ✎

2a. If $\tan\theta < 0$ and $\cos\theta < 0$, name the quadrant in which θ lies.

2b. Given that $\tan\theta = -\frac{2}{3}$ and $\sin\theta > 0$, find $\cos\theta$ and $\csc\theta$.

Objective #3: Find reference angles.

✔ ***Solved Problem #3***	✎ ***Pencil Problem #3***
3a. Find the reference angle for $\theta = 210°$. The angle lies in quadrant III. The reference angle is $\theta' = 210° - 180° = 30°$.	**3a.** Find the reference angle for $\theta = 160°$.
3b. Find the reference angle for $\theta = \frac{7\pi}{4}$. The angle lies in quadrant IV. The reference angle is $\theta' = 2\pi - \frac{7\pi}{4} = \frac{8\pi}{4} - \frac{7\pi}{4} = \frac{\pi}{4}$.	**3b.** Find the reference angle for $\theta = \frac{5\pi}{6}$.
3c. Find the reference angle for $\theta = -240°$. The angle lies in quadrant II. The positive acute angle formed by the terminal side of θ and the x-axis is 60°. The reference angle is $\theta' = 60°$.	**3c.** Find the reference angle for $\theta = -335°$.
3d. Find the reference angle for $\theta = 665°$. Subtract 360° to find a positive coterminal angle less than 360°: 665° – 360° = 305°. The angle $\alpha = 305°$ lies in quadrant IV. The reference angle is $\alpha' = 360° - 305° = 55°$.	**3d.** Find the reference angle for $\theta = 565°$.

3e. Find the reference angle for $\theta = -\frac{11\pi}{3}$.

Add 4π to find a positive coterminal angle less than 2π: $-\frac{11\pi}{3}+4\pi = -\frac{11\pi}{3}+\frac{12\pi}{3}=\frac{\pi}{3}$.

The angle $\alpha = \frac{\pi}{3}$ lies in quadrant I. The reference angle is $\alpha' = \frac{\pi}{3}$.

3e. Find the reference angle for $\theta = -\frac{11\pi}{4}$.

Objective #4: Use reference angles to evaluate trigonometric functions.

✔ *Solved Problem #4*

4a. Use a reference angle to find the exact value of $\sin 300°$.

A 300° angle lies in quadrant IV, where the sine function is negative. The reference angle is $\theta' = 360° - 300° = 60°$.

$$\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

4b. Use a reference angle to find the exact value of $\tan\frac{5\pi}{4}$.

A $\frac{5\pi}{4}$ angle lies in quadrant III, where the tangent function is positive. The reference angle is $\theta' = \frac{5\pi}{4}-\pi = \frac{5\pi}{4}-\frac{4\pi}{4}=\frac{\pi}{4}$.

$$\tan\frac{5\pi}{4} = +\tan\frac{\pi}{4} = 1$$

✎ *Pencil Problem #4* ✐

4a. Use a reference angle to find the exact value of $\cos 225°$.

4b. Use a reference angle to find the exact value of $\sin\frac{2\pi}{3}$.

4c. Use a reference angle to find the exact value of $\sec\left(-\frac{\pi}{6}\right)$.

A $-\frac{\pi}{6}$ angle lies in quadrant IV, where the secant function is positive. Furthermore, a $-\frac{\pi}{6}$ angle forms an acute of $\frac{\pi}{6}$ with the *x*-axis. The reference angle is $\theta' = \frac{\pi}{6}$.

$$\sec\left(-\frac{\pi}{6}\right) = +\sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}$$

4c. Use a reference angle to find the exact value of $\tan\left(-\frac{\pi}{4}\right)$.

4d. Use a reference angle to find the exact value of $\cos\frac{17\pi}{6}$.

Subtract 2π to find a positive coterminal angle less than 2π. $\frac{17\pi}{6} - 2\pi = \frac{17\pi}{6} - \frac{12\pi}{6} = \frac{5\pi}{6}$.

A $\frac{5\pi}{6}$ angle lies in quadrant II, where the cosine function is negative. The reference angle is $\theta' = \pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}$.

$$\cos\frac{17\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

4d. Use a reference angle to find the exact value of $\cot\frac{19\pi}{6}$.

<u>Answers</u> <u>for Pencil Problems *(Textbook Exercise references in parentheses)*</u>:

1a. $\sin\theta = -\frac{5\sqrt{29}}{29},\ \cos\theta = -\frac{2\sqrt{29}}{29},\ \tan\theta = \frac{5}{2},\ \csc\theta = -\frac{\sqrt{29}}{5},\ \sec\theta = -\frac{\sqrt{29}}{2},\ \cot\theta = \frac{2}{5}$ *(4.4 #7)*

1b. undefined *(4.4 #13)*

2a. quadrant II *(4.4 #21)* **2b.** $\cos\theta = -\frac{3\sqrt{13}}{13},\ \csc\theta = \frac{\sqrt{13}}{2}$ *(4.4 #29)*

3a. 20° *(4.4 #35)* **3b.** $\frac{\pi}{6}$ *(4.4 #43)* **3c.** 25° *(4.4 #47)* **3d.** 25° *(4.4 #51)* **3e.** $\frac{\pi}{4}$ *(4.4 #57)*

4a. $-\frac{\sqrt{2}}{2}$ *(4.4 #61)* **4b.** $\frac{\sqrt{3}}{2}$ *(4.4 #67)* **4c.** -1 *(4.4 #75)* **4d.** $\sqrt{3}$ *(4.4 #79)*

Section 4.5
Graphs of Sine and Cosine Functions

The Graph That Looks Like a Rollercoaster

In this section, we study the graphs of the sine and cosine functions. The ups and downs of the graphs may remind you of a rollercoaster. The graphs rise to peaks at maximum points then plunge to minimum points where they promptly change direction and rise again.

However, unlike a rollercoaster ride, the ups and downs of these graphs go on forever. The shapes of these graphs and their repetitive properties make it even more obvious why trigonometric functions are used to model cyclic behavior.

Objective #1: Understand the graph of $y = \sin x$.

✔ *Solved Problem #1*

1. True or false: The graph of $y = \sin x$ is symmetric with respect to the *y*-axis.

False; the sine function is an odd function and is symmetric with respect to the origin but not the *y*-axis.

✎ *Pencil Problem #1* ✎

1. True or false: The graph of $y = \sin x$ has no gaps or holes and extends indefinitely in both directions.

Objective #2: Graph variations of $y = \sin x$.

✔ *Solved Problem #2*

2a. Determine the amplitude and the period of $y = 2\sin\frac{1}{2}x$. Then graph the function for $0 \le x \le 8\pi$.

Comparing $y = 2\sin\frac{1}{2}x$ to $y = A\sin Bx$, we see that $A = 2$ and $B = \frac{1}{2}$.

Amplitude: $|A| = |2| = 2$

Period: $\frac{2\pi}{B} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

The amplitude tells us that the maximum value of the function is 2 and the minimum value is –2. The period tells us that the graph completes one cycle between 0 and 4π.

(continued on next page)

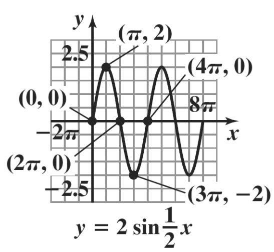
✎ *Pencil Problem #2* ✎

2a. Determine the amplitude and the period of $y = 3\sin\frac{1}{2}x$. Then graph one period of the function.

Divide the period by 4: $\frac{4\pi}{4} = \pi$. The x-values of the five key points are $x_1 = 0$, $x_2 = 0 + \pi = \pi$, $x_3 = \pi + \pi = 2\pi$, $x_4 = 2\pi + \pi = 3\pi$, and $x_5 = 3\pi + \pi = 4\pi$. Now find the value of y for each of these x-values.

$$y = 2\sin\left(\frac{1}{2}\cdot 0\right) = 2\sin 0 = 2\cdot 0 = 0 : (0,\ 0)$$
$$y = 2\sin\left(\frac{1}{2}\cdot \pi\right) = 2\sin\frac{\pi}{2} = 2\cdot 1 = 2 : (\pi,\ 2)$$
$$y = 2\sin\left(\frac{1}{2}\cdot 2\pi\right) = 2\sin \pi = 2\cdot 0 = 0 : (2\pi,\ 0)$$
$$y = 2\sin\left(\frac{1}{2}\cdot 3\pi\right) = 2\sin\frac{3\pi}{2} = 2(-1) = -2 : (3\pi,\ -2)$$
$$y = 2\sin\left(\frac{1}{2}\cdot 4\pi\right) = 2\sin 2\pi = 2\cdot 0 = 0 : (4\pi,\ 0)$$

Notice the pattern: x-intercept, maximum, x-intercept, minimum, x-intercept. Plot these points and connect them with a smooth curve. Extend the graph one period to the right in order to graph from $0 \le x \le 8\pi$.



2b. Determine the amplitude, period, and phase shift of $y=3\sin\left(2x-\frac{\pi}{3}\right)$. Then graph one period of the function.

Comparing $y=3\sin\left(2x-\frac{\pi}{3}\right)$ to $y=A\sin(Bx-C)$, we see that $A = 3$, $B = 2$, and $C = \frac{\pi}{3}$.

Amplitude: $|A|=|3|=3$

Period: $\frac{2\pi}{B}=\frac{2\pi}{2}=\pi$

Phase shift: $\frac{C}{B}=\frac{\frac{\pi}{3}}{2}=\frac{\pi}{3}\cdot\frac{1}{2}=\frac{\pi}{6}$

The amplitude tells us that the maximum value of the function is 3 and the minimum value is −3. The period tells us that each cycle is of length π. The phase shift tells us that a cycle starts at $\frac{\pi}{6}$.

Divide the period by 4: $\frac{\pi}{4}$. The x-values of the five key points are $x_1=\frac{\pi}{6}$,

$x_2=\frac{\pi}{6}+\frac{\pi}{4}=\frac{2\pi}{12}+\frac{3\pi}{12}=\frac{5\pi}{12}$,

$x_3=\frac{5\pi}{12}+\frac{\pi}{4}=\frac{5\pi}{12}+\frac{3\pi}{12}=\frac{8\pi}{12}=\frac{2\pi}{3}$,

$x_4=\frac{2\pi}{3}+\frac{\pi}{4}=\frac{8\pi}{12}+\frac{3\pi}{12}=\frac{11\pi}{12}$, and

$x_5=\frac{11\pi}{12}+\frac{\pi}{4}=\frac{11\pi}{12}+\frac{3\pi}{12}=\frac{14\pi}{12}=\frac{7\pi}{6}$. Now find the value of y for each of these x-values.

$$y=3\sin\left(2\cdot\frac{\pi}{6}-\frac{\pi}{3}\right)=3\sin 0=0:\left(\frac{\pi}{6},\ 0\right)$$

$$y=3\sin\left(2\cdot\frac{5\pi}{12}-\frac{\pi}{3}\right)=3\sin\frac{\pi}{2}=3:\left(\frac{5\pi}{12},\ 3\right)$$

$$y=3\sin\left(2\cdot\frac{2\pi}{3}-\frac{\pi}{3}\right)=3\sin\pi=0:\left(\frac{2\pi}{3},\ 0\right)$$

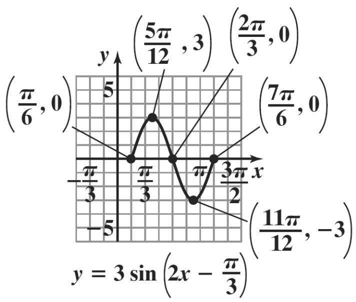
$$y=3\sin\left(2\cdot\frac{11\pi}{12}-\frac{\pi}{3}\right)=3\sin\frac{3\pi}{2}=-3:\left(\frac{11\pi}{12},\ -3\right)$$

$$y=3\sin\left(2\cdot\frac{7\pi}{6}-\frac{\pi}{3}\right)=3\sin 2\pi=0:\left(\frac{7\pi}{6},\ 0\right)$$

(continued on next page)

2b. Determine the amplitude, period, and phase shift of $y=3\sin(2x-\pi)$. Then graph one period of the function.

Notice the pattern: x-intercept, maximum, x-intercept, minimum, x-intercept. Plot these points and connect them with a smooth curve.



Objective #3: Understand the graph of $y = \cos x$..

✔ *Solved Problem #3*

3. True or false: The graph of $y = \cos x$ is symmetric with respect to the y-axis.

True; the cosine function is an even function, and the graphs of even functions have symmetry with respect to the y-axis.

✎ *Pencil Problem #3* ✎

3. True or false: The graph of $y = \cos x$ illustrates that the range of the cosine function is $(-\infty, \infty)$.

Objective #4: Graph variations of $y = \cos x$.

✔ *Solved Problem #4*

4. Determine the amplitude, period, and phase shift of $y = \frac{3}{2}\cos(2x + \pi)$. Then graph one period of the function.

Comparing $y = \frac{3}{2}\cos(2x + \pi)$ to $y = A\sin(Bx - C)$, we see that $A = \frac{3}{2}$, $B = 2$, and $C = -\pi$.

Amplitude: $|A| = \left|\frac{3}{2}\right| = \frac{3}{2}$

Period: $\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase shift: $\frac{C}{B} = \frac{-\pi}{2} = -\frac{\pi}{2}$

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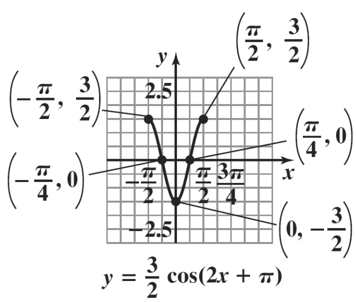
✎ *Pencil Problem #4* ✎

4. Determine the amplitude, period, and phase shift of $y = \frac{1}{2}\cos\left(3x + \frac{\pi}{2}\right)$. Then graph one period of the function.

The amplitude tells us that the maximum value of the function is $\frac{3}{2}$ and the minimum value is $-\frac{3}{2}$. The period tells us that each cycle is of length π. The phase shift tells us that a cycle starts at $-\frac{\pi}{2}$.

Divide the period by 4: $\frac{\pi}{4}$. The x-values of the five key points are $x_1 = -\frac{\pi}{2}$,
$x_2 = -\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2\pi}{4} + \frac{\pi}{4} = -\frac{\pi}{4}$, $x_3 = -\frac{\pi}{4} + \frac{\pi}{4} = 0$,
$x_4 = 0 + \frac{\pi}{4} = \frac{\pi}{4}$, and $x_5 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$. Now find the value of y for each of these x-values by evaluating the function. The five key points are $\left(-\frac{\pi}{2}, \frac{3}{2}\right)$, $\left(-\frac{\pi}{4}, 0\right)$, $\left(0, -\frac{3}{2}\right)$, $\left(\frac{\pi}{4}, 0\right)$, and $\left(\frac{\pi}{2}, \frac{3}{2}\right)$. Notice the pattern: maximum, x-intercept, minimum, x-intercept, maximum. Plot these points and connect them with a smooth curve.



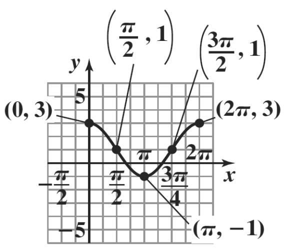
Objective #5: Use vertical shifts of sine and cosine curves.

✔ *Solved Problem #5*

5. Graph one period of the function $y = 2\cos x + 1$.

The graph of $y = 2\cos x + 1$ is the graph of $y = 2\cos x$ shifted one unit up. The amplitude is 2 and the period is 2π. The graph oscillates 2 units below and 2 units above the line $y = 1$.

The x-values for the five key points are 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π. The five key points are (0, 3), $\left(\frac{\pi}{2}, 1\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 1\right)$, and $(2\pi, 3)$. Plot these points and connect them with a smooth curve.



$y = 2\cos x + 1$

✎ *Pencil Problem #5* ✎

5. Graph one period of the function $y = \sin x + 2$.

Objective #6: Model periodic behavior.

✔ *Solved Problem #6*

6. A region that is 30° north of the Equator averages a minimum of 10 hours of daylight in December. Hours of daylight are at a maximum of 14 hours in June. Let x represent the month of the year, with 1 for January, 2 for February, 3 for March, and 12 for December. If y represents the number of hours of daylight in month x, use a sine function of the form $y = A\sin(Bx - C) + D$ to model the hours of daylight.

We need to determine values for A, B, C, and D in $y = A\sin(Bx - C) + D$.

To find D, notice that the values range between a minimum of 10 and a maximum of 14. The middle value is 12, so $D = 12$.

To find A, notice that the minimum, 10, and maximum, 14, are 2 units below and above the middle value, $D = 12$. The amplitude is 2, so $A = 2$.

To find B, notice that the period is 12 months (one year). Use the period formula and solve for B.

$$\frac{2\pi}{B} = 12$$
$$2\pi = 12B$$
$$B = \frac{2\pi}{12} = \frac{\pi}{6}$$

To find C, notice that a middle value occurs in March ($x = 3$), so we can begin a cycle in March. Use the phase shift formula with this value of x and the value of B just found.

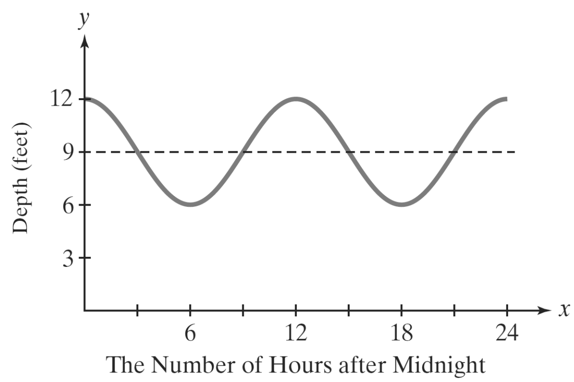
$$Bx - C = 0$$
$$\frac{\pi}{6} \cdot 3 - C = 0$$
$$\frac{\pi}{2} = C$$

Substitute the values for A, B, C, and D into $y = A\sin(Bx - C) + D.$ The model is

$$y = 2\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 12.$$

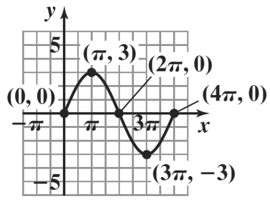
✎ *Pencil Problem #6* ✎

6. The figure shows the depth of water at the end of a boat dock. The depth is 6 feet at low tide and 12 feet at high tide. On a certain day, low tide occurs at 6 a.m. and high tide occurs at noon. If y represents the depth of the water x hours after midnight, use a cosine function of the form $y = A\cos Bx + D$ to model the water's depth.

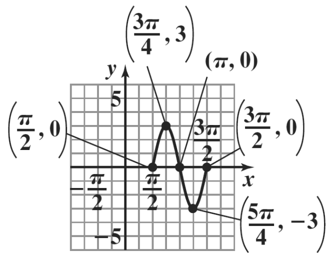


Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. true

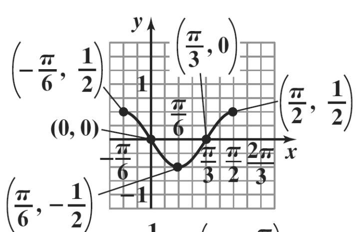


2a. $y = 3\sin\frac{1}{2}x$ Amplitude: 3; period: 4π *(4.5 #9)*

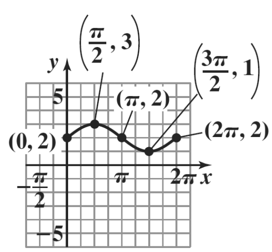


2b. $y = 3\sin(2x - \pi)$ Amplitude: 3; period: π; phase shift: $\frac{\pi}{2}$ *(4.5 #21)*

3. false



4a. $y = \frac{1}{2}\cos\left(3x + \frac{\pi}{2}\right)$ Amplitude: $\frac{1}{2}$; period: $\frac{2\pi}{3}$; phase shift: $-\frac{\pi}{6}$ *(4.5 #47)*



5. $y = \sin x + 2$ *(4.5 #53)*

6. $y = 3\cos\frac{\pi x}{6} + 9$ *(4.5 #87)*

Section 4.6
Graphs of Other Trigonometric Functions

Trig Functions Without Bounds

We have seen that sine and cosine functions can be used to model phenomena that are cyclic in nature, such as the number of daylight hours in a day at a specific location over a period of years. But it's not just the periodic properties of sine and cosine that are important in these situations; it's also their limited ranges. The fact that the ranges of sine and cosine functions are bounded makes them suitable for modeling phenomena that never exceed certain minimum and maximum values.

In this section, we will see that the remaining trigonometric functions, while periodic like sine and cosine, do not have bounded ranges like sine and cosine. These functions are more suitable for describing cyclic phenomena that do not have natural limits, such as the location where a beam of light from a rotating source strikes a flat surface.

Objective #1: Understand the graph of $y = \tan x$.

✔ Solved Problem #1

1. True or false: The graph of $y = \tan x$ is symmetric with respect to the y-axis.

False; the tangent function is an odd function and is symmetric with respect to the origin but not the y-axis.

✎ Pencil Problem #1 ✎

1. True or false: The graph of $y = \tan x$ has no gaps or holes and extends indefinitely in both directions.

Objective #2: Graph variations of $y = \tan x$.

✔ Solved Problem #2

2a. Graph $y = 3\tan 2x$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$.

Find two consecutive asymptotes by solving $-\frac{\pi}{2} < Bx - C < \frac{\pi}{2}$. In this case, $Bx - C = 2x$.

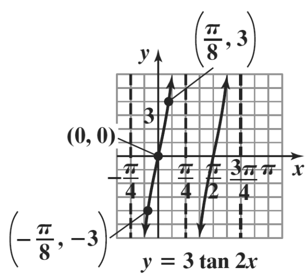
$$-\frac{\pi}{2} < 2x < \frac{\pi}{2}$$
$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$

(continued on next page)

✎ Pencil Problem #2 ✎

2a. Graph two full periods of $y = \frac{1}{2}\tan 2x$.

The graph completes one cycle on the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and has consecutive vertical asymptotes at $x=-\frac{\pi}{4}$ and $x=\frac{\pi}{4}$. The x-intercept is midway between the asymptotes at (0, 0). The x-values $-\frac{\pi}{8}$ and $\frac{\pi}{8}$ are $\frac{1}{4}$ and $\frac{3}{4}$ of the way between the asymptotes; the points $\left(-\frac{\pi}{8}, -3\right)$ and $\left(\frac{\pi}{8}, 3\right)$ are on the graph. Plot these three points and the asymptotes. Graph one period of the function by drawing a smooth curve through the points and approaching the asymptotes. Complete one more period to the right in order to show the graph for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$.



2b. Graph two full periods of $y=\tan\left(x-\frac{\pi}{2}\right)$.

Find two consecutive asymptotes by solving $-\frac{\pi}{2} < Bx - C < \frac{\pi}{2}$. In this case, $Bx - C = x - \frac{\pi}{2}$.

$$-\frac{\pi}{2} < x - \frac{\pi}{2} < \frac{\pi}{2}$$
$$-\frac{\pi}{2} + \frac{\pi}{2} < x < \frac{\pi}{2} + \frac{\pi}{2}$$
$$0 < x < \pi$$

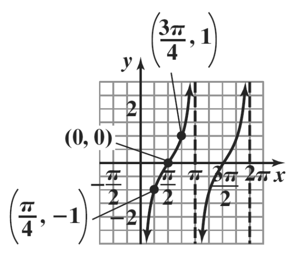
(continued on next page)

2b. Graph two full periods of $y=\tan(x-\pi)$.

The graph completes one cycle on the interval $(0, \pi)$ and has consecutive vertical asymptotes at $x = 0$ and $x = \pi$. The x-intercept is midway between the asymptotes at $\left(\frac{\pi}{2}, 0\right)$. The x-values $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are $\frac{1}{4}$ and $\frac{3}{4}$ of the way between the asymptotes; the points $\left(\frac{\pi}{4}, -1\right)$ and $\left(\frac{3\pi}{4}, 1\right)$ are on the graph.

Plot these three points and the asymptotes. Graph one period of the function by drawing a smooth curve through the points and approaching the asymptotes. Complete one more period to the left or to the right in order to show two full periods.

The graph of $y = \tan\left(x - \frac{\pi}{2}\right)$ is the graph of $y = \tan x$ shifted $\frac{\pi}{2}$ units to the right.



$y = \tan\left(x - \frac{\pi}{2}\right)$

Objective #3: Understand the graph of $y = \cot x$.

✔ Solved Problem #3

3. True or false: The graph of $y = \cot x$ has asymptotes at values of x where the graph of $y = \tan x$ has x-intercepts.

True; the cotangent function is undefined at values of x for which tangent is 0.

Pencil Problem #3

3. True or false: The graph of $y = \cot x$ illustrates that the range of the cotangent function is $(-\infty, \infty)$.

Objective #4: Graph variations of $y = \cot x$.

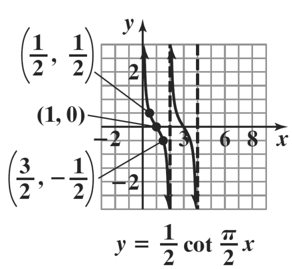
✔ Solved Problem #4

4. Graph $y = \frac{1}{2}\cot\frac{\pi}{2}x$.

Find two consecutive asymptotes by solving $0 < Bx - C < \pi$. In this case, $Bx - C = \frac{\pi}{2}x$.

$$0 < \frac{\pi}{2}x < \pi$$
$$\frac{2}{\pi}\cdot 0 < x < \frac{2}{\pi}\cdot\pi$$
$$0 < x < 2$$

The graph completes one cycle on the interval (0, 2) and has consecutive vertical asymptotes at $x = 0$ and $x = 2$. The x-intercept is midway between the asymptotes at (1, 0). The x-values $\frac{1}{2}$ and $\frac{3}{2}$ are $\frac{1}{4}$ and $\frac{3}{4}$ of the way between the asymptotes; the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{3}{2}, -\frac{1}{2}\right)$ are on the graph. Plot these three points and the asymptotes. Graph one period of the function by drawing a smooth curve through the points and approaching the asymptotes.



✎ Pencil Problem #4 ✎

4. Graph two periods of $y = \frac{1}{2}\cot 2x$.

Objective #5: Understand the graphs of $y = \csc x$ and $y = \sec x$.

✔ *Solved Problem #5*

5. True or false: The graph of $y = \csc x$ illustrates that the range of the cosecant function is $(-\infty, \infty)$.

False; the range of the cosecant function is $(-\infty, -1] \cup [1, \infty)$. It does not include values of y between -1 and 1.

✎ *Pencil Problem #5* ✎

5. True or false: The graph of $y = \sec x$ has asymptotes at values of x where the graph of $y = \sin x$ has x-intercepts.

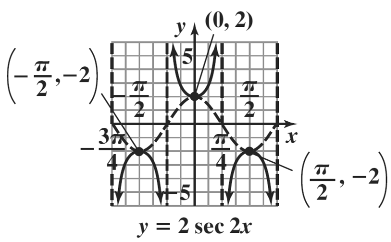
Objective #6: Graph variations of $y = \csc x$ and $y = \sec x$.

✔ *Solved Problem #6*

6. Graph $y = 2\sec 2x$ for $-\frac{3\pi}{4} < x < \frac{3\pi}{4}$.

Begin by graphing $y = 2\cos 2x$ where secant has been replaced by cosine, its reciprocal function. The amplitude of $y = 2\cos 2x$ is 2 and the period is $\frac{2\pi}{2} = \pi$. Note that for $-\frac{3\pi}{4} < x < \frac{3\pi}{4}$ the graph of $y = 2\cos 2x$ has x-intercepts at $x = -\frac{3\pi}{4}$, $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$, and $x = \frac{3\pi}{4}$. The graph of $y = 2\cos 2x$ attains its minimum value of -2 at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ and its maximum value of 2 at $x = 0$.

At each x-intercept of the graph of $y = 2\cos 2x$ draw a vertical asymptote. Graph $y = 2\sec 2x$ by starting at each minimum or maximum point and approaching the asymptotes in each direction.



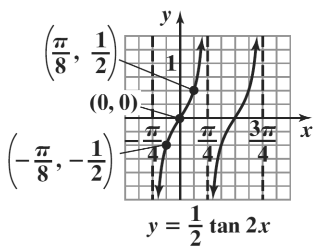
✎ *Pencil Problem #6* ✎

6. Graph two periods of $y = \frac{1}{2}\csc\frac{x}{2}$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

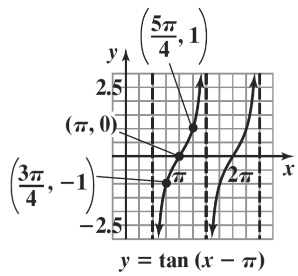
1. false

2a.



(4.6 #7)

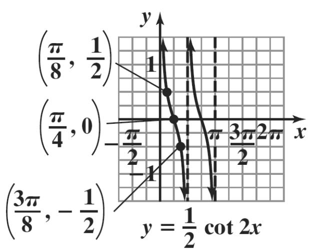
2b.



(4.6 #11)

3. true

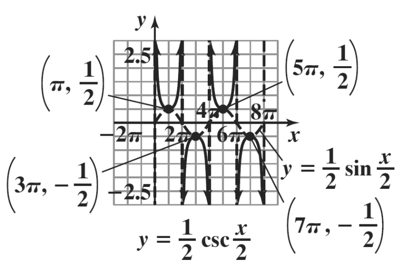
4.



(4.6 #19)

5. false

6.



(4.6 #31)

Section 4.7
Inverse Trigonometric Functions

For Your Viewing Pleasure

Your total movie experience at your local cinema is affected by many variables, including your distance from the screen. If you sit too close, your viewing angle is too small. If you sit too far back, the image is too small.

In this section's Exercise Set, you will see how inverse trigonometric functions can be used to model your viewing angle in terms of your distance from the screen and help you find the seat that will optimize your viewing pleasure.

Objective #1: Understand and use the inverse sine function.

✔ *Solved Problem #1*

1a. Find the exact value of $\sin^{-1}\frac{\sqrt{3}}{2}$.

Let $\theta = \sin^{-1}\frac{\sqrt{3}}{2}$. Then $\sin\theta = \frac{\sqrt{3}}{2}$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

We must find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{3}}{2}$. Using a table of values for the sine function for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we see that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Thus, $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

✎ *Pencil Problem #1* ✎

1a. Find the exact value of $\sin^{-1}\frac{1}{2}$.

1b. Find the exact value of $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Let $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Then $\sin\theta = -\frac{\sqrt{2}}{2}$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

We must find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{2}}{2}$. Using a table of values for the sine function for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we see that $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Thus, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

1b. Find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Objective #2: Understand and use the inverse cosine function

✔ *Solved Problem #2*

2. Find the exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

Let $\theta = \cos^{-1}\left(-\frac{1}{2}\right)$.

Then $\cos\theta = -\frac{1}{2}$ where $0 \le \theta \le \pi$.

We must find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{1}{2}$. Using a table of values for the cosine function for $0 \le \theta \le \pi$, we see that $\cos\frac{2\pi}{3} = -\frac{1}{2}$.

Thus, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

✎ *Pencil Problem #2* ✐

2. Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Objective #3: Understand and use the inverse tangent function.

✔ *Solved Problem #3*

3. Find the exact value of $\tan^{-1}(-1)$.

Let $\theta = \tan^{-1}(-1)$.

Then $\tan\theta = -1$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

We must find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals –1. Using a table of values for the tangent function for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that $\tan\left(-\frac{\pi}{4}\right) = -1$.

Thus, $\tan^{-1}(-1) = -\frac{\pi}{4}$.

✎ *Pencil Problem #3* ✎

3. Find the exact value of $\tan^{-1} 0$.

Objective #4: Use a calculator to evaluate inverse trigonometric functions.

✔ *Solved Problem #4*

4a. Use a calculator to find the value of $\cos^{-1}\frac{1}{3}$ to four decimal places.

To access the inverse cosine function, you will need to press the 2nd function key and then the COS key. Use radian mode.

Scientific calculator: $1 \div 3 =$ 2nd COS
Graphing calculator: 2nd COS $(1 \div 3)$ ENTER

The display should read 1.2310, rounded to four places.

✎ *Pencil Problem #4* ✎

4a. Use a calculator to find the value of $\cos^{-1}\frac{3}{8}$ to two decimal places.

4b. Use a calculator to find the value of $\tan^{-1}(-35.85)$ to four decimal places.

To access the inverse tangent function, you will need to press the 2nd function key and then the TAN key.

Use radian mode.

Scientific calculator: 35.85 +/– 2nd TAN
Graphing calculator: 2nd TAN (–) 35.85 ENTER

The display should read −1.5429, rounded to four places.

4b. Use a calculator to find the value of $\sin^{-1}(-0.32)$ to two decimal places.

Objective #5: Find exact values of composite functions with inverse trigonometric functions.

✔ *Solved Problem #5*

5a. Find the exact value, if possible: $\cos(\cos^{-1} 0.7)$.

Since 0.7 is in the interval [−1, 1], we can use the inverse property $\cos(\cos^{-1} x) = x$.

$\cos(\cos^{-1} 0.7) = 0.7$

5b. Find the exact value, if possible: $\sin^{-1}(\sin \pi)$.

Since π is not in the $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we cannot use the inverse property $\sin^{-1}(\sin x) = x$. We first evaluate $\sin \pi = 0$, and then evaluate $\sin^{-1} 0$.

$\sin^{-1}(\sin \pi) = \sin^{-1} 0 = 0$

✎ *Pencil Problem #5* ✐

5a. Find the exact value, if possible: $\sin(\sin^{-1} 0.9)$.

5b. Find the exact value, if possible: $\sin^{-1}\left(\sin \frac{5\pi}{6}\right)$.

5c. Find the exact value, if possible: $\cos[\cos^{-1}(-1.2)]$.

Since -1.2 is not in the domain of the inverse cosine function, $[-1, 1]$, $\cos[\cos^{-1}(-1.2)]$ is not defined.

5c. Find the exact value, if possible: $\sin(\sin^{-1}\pi)$.

5d. Find the exact value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Let θ represent the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $\frac{3}{4}$. Thus, $\theta = \tan^{-1}\frac{3}{4}$.

So, $\tan\theta = \frac{3}{4}$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Since $\tan\theta$ is positive, θ must be in $\left(0, \frac{\pi}{2}\right)$. Thus, θ lies in quadrant I, where both x and y are positive.

$\tan\theta = \frac{3}{4} = \frac{y}{x}$, so $x = 4$ and $y = 3$

Find r and then use the value of r to find $\sin\theta$.

$r = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\sin\theta = \frac{y}{r} = \frac{3}{5}$

Thus, $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\theta = \frac{3}{5}$.

5d. Find the exact value of $\cos\left(\sin^{-1}\frac{4}{5}\right)$.

5e. Find the exact value of $\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Let θ represent the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{1}{2}$. Thus, $\theta = \sin^{-1}\left(-\frac{1}{2}\right)$.

So, $\sin\theta = -\frac{1}{2}$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Since $\sin\theta$ is negative, θ must be in $\left[-\frac{\pi}{2}, 0\right)$. Thus, θ lies in quadrant IV, where both x is positive and y is negative.

$\sin\theta = -\frac{1}{2} = \frac{y}{r} = \frac{-1}{r}$, so $y = -1$ and $r = 2$

Find x and then use the value of x to find $\cos\theta$.

$$r = \sqrt{x^2 + y^2}$$
$$2 = \sqrt{x^2 + (-1)^2}$$
$$4 = x^2 + 1$$
$$3 = x^2$$
$$x = \sqrt{3}, \text{ since } x > 0$$
$$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$$

Thus, $\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \cos\theta = \frac{\sqrt{3}}{2}$.

In this problem, we know how to find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right)$, so we could have also proceeded as follows:

$$\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

5e. Find the exact value of $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$.

5f. If $x > 0$, write $\sec(\tan^{-1} x)$ as an algebraic expression in x.

Let θ represent the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

Thus, $\theta = \tan^{-1} x$ and $\tan\theta = x$, where $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Because $x > 0$, $\tan\theta$ is positive. Thus, θ is a first-quadrant angle.

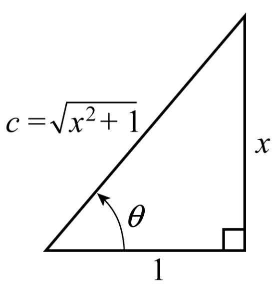
Draw a right triangle and label one of the acute angles θ. Since

$$\tan\theta = x = \frac{x}{1} = \frac{\text{opposite}}{\text{adjacent}},$$

the length of the side opposite θ is x and the length of the adjacent side is 1.

Use the Pythagorean theorem to find the hypotenuse.

$$c^2 = x^2 + 1^2$$
$$c = \sqrt{x^2+1}$$



Thus, $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{x^2+1}}{1} = \sqrt{x^2+1}$ and consequently,

$$\sec(\tan^{-1} x) = \sec\theta = \sqrt{x^2+1}.$$

5f. If $x > 0$, write $\tan(\cos^{-1} x)$ as an algebraic expression in x.

<u>Answers</u> <u>for Pencil Problems</u> *<u>(Textbook Exercise references in parentheses)</u>*:

1a. $\dfrac{\pi}{6}$ *(4.7 #1)* **1b.** $-\dfrac{\pi}{6}$ *(4.7 #5)*

2. $\dfrac{3\pi}{4}$ *(4.7 #9)*

3. 0 *(4.7 #15)*

4a. 1.19 *(4.7 #23)* **4b.** –0.33 *(4.7 #21)*

5a. 0.9 *(4.7 #31)* **5b.** $\dfrac{\pi}{6}$ *(4.7 #35)* **5c.** undefined *(4.7 #45)* **5d.** $\dfrac{3}{5}$ *(4.7 #47)*

5e. 2 *(4.7 #59)* **5f.** $\dfrac{\sqrt{1-x^2}}{x}$ *(4.7 #63)*

Section 4.8
Applications of Trigonometric Functions

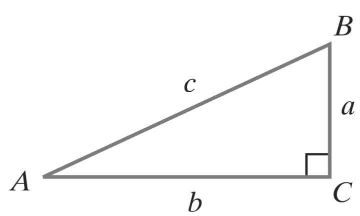
Up, Up, & Away, Around the Corner, and Back & Forth

From finding the heights of tall buildings to modeling cyclic behavior, trigonometry is very useful. In this section, you will see how finding missing parts of right triangles has many practical applications, how ships at sea can be located using bearings and trigonometry, and even how the motion of a ball attached to a spring can be described using a sinusoidal function.

Objective #1: Solve a right triangle.

✔ *Solved Problem #1*

1a. Let $A = 62.7°$ and $a = 8.4$ in the triangle shown below. Solve the right triangle, rounding lengths to two decimal places.



To solve the triangle, we need to find B, b, and c.

We begin with B. We know that $A + B = 90°$.

$$62.7° + B = 90°$$
$$B = 90° - 62.7° = 27.3°$$

Next we find b. Note that b is opposite $B = 27.3°$ and $a = 8.4$ is adjacent to $B = 27.3°$. We set up an equation relating a, b, and B using the tangent function. Then we solve for b.

$$\tan B = \frac{\text{side opposite } B}{\text{side adjacent to } B} = \frac{b}{a}$$
$$\tan 27.3° = \frac{b}{8.4}$$
$$b = 8.4 \tan 27.3° \approx 4.34$$

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✎ *Pencil Problem #1* ✎

1a. Let $A = 23.5°$ and $b = 10$ in the triangle shown in Solved Problem #1a. Solve the right triangle, rounding lengths to two decimal places.

When finding c, we choose not to use the value of b just found because it was rounded. We will again use $B = 27.3°$ and $a = 8.4$. Since a is adjacent to B and we are finding the hypotenuse, we use the cosine function.

$$\cos B = \frac{\text{side adjacent to } B}{\text{hypotenuse}} = \frac{a}{c}$$

$$\cos 27.3° = \frac{8.4}{c}$$

$$c \cos 27.3° = 8.4$$

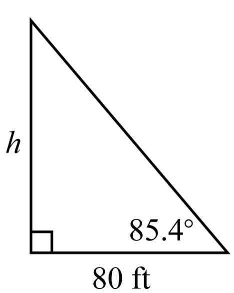
$$c = \frac{8.4}{\cos 27.3°} \approx 9.45$$

Thus, $B = 27.3°$, $b \approx 4.34$, and $c \approx 9.45$.

Note that angle A could have been used in the calculations to find b and c, $\tan A = \frac{a}{b}$ and $\sin A = \frac{a}{c}$, without using rounded values.

1b. From a point on level ground 80 feet from the base of a tower, the angle of elevation is 85.4°. Approximate the height of the tower to the nearest foot.

We draw a right triangle to illustrate the situation. The tower and the ground form the right angle. One of the acute angles measures 85.4°, and the side adjacent to it is 80 feet. The tower is opposite the 85.4° angle; we let h represent the height of the tower.



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1b. From a point on level ground 5280 feet (1 mile) from the base of a television transmitting tower, the angle of elevation is 21.3°. Approximate the height of the tower to the nearest foot.

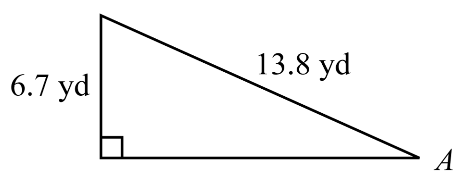
Since we are looking for the side opposite the known acute angle and we know the side adjacent to that angle, we will use tangent.

$$\tan 85.4° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{h}{80}$$
$$h = 80\tan 85.4° \approx 994$$

The tower is approximately 994 feet tall.

1c. A guy wire is 13.8 yards long and is attached from the ground to a pole at a point 6.7 yards above the ground. Find the angle, to the nearest tenth, that the wire makes with the ground.

We draw a right triangle to illustrate the situation. The pole and the ground form the right angle. We are looking for the angle opposite the pole; call it *A*. The side opposite *a* is 6.7 yards, and the hypotenuse is 13.8 yards.



Since we know the side opposite *A* and the hypotenuse, we will use the sine function.

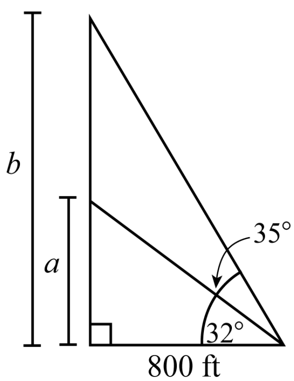
$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{6.7}{13.8}$$
$$A = \sin^{-1}\frac{6.7}{13.8} \approx 29.0°$$

The wire makes an angle of approximately 29.0° with the ground.

1c. A wheelchair ramp is built beside the steps to the campus library. Find the angle of elevation of the 23-foot ramp, to the nearest tenth of a degree, if its final height is 6 feet.

1d. You are standing on level ground 800 feet from Mt. Rushmore, looking at the sculpture of Abraham Lincoln's face. The angle of elevation to the bottom of the sculpture is 32°, and the angle of elevation to the top is 35°. Find the height of the sculpture of Lincoln's face to the nearest tenth of a foot.

Refer to the figure below, where a represents the distance to the bottom of the sculpture and b represents the distance to the top of the sculpture. The height of the sculpture is then $b - a$. We need to find a and b and then subtract.



Since in each case we are finding the length of the leg opposite a known angle and we also know the length of the leg adjacent to the known angle, we will use the tangent function twice.

$$\tan 32° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{800}$$
$$a = 800 \tan 32° \approx 499.9$$

$$\tan 35° = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{b}{800}$$
$$b = 800 \tan 35° \approx 560.2$$

$$b - a = 560.2 - 499.9 = 60.3$$

Lincoln's face is approximately 60.3 feet tall.

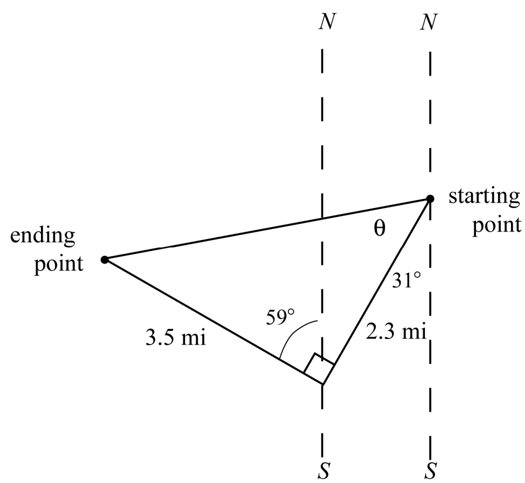
1d. A hot-air balloon is rising vertically. From a point on level ground 125 feet from the point directly under the passenger compartment, the angle of elevation changes from 19.2° to 31.7°. How far, to the nearest tenth of a foot, does the balloon rise during this period?

Objective #2: Solve problems involving bearings.

✔ *Solved Problem #2*

2. You hike 2.3 miles on a bearing of S 31° W. Then you turn 90° clockwise and hike 3.5 miles on a bearing of N 59° W. At that time, what is your bearing, to the nearest tenth of a degree, from your starting point?

The figure below illustrates the situation. Notice that we have formed a triangle by drawing a segment from the starting point to the ending point. The triangle is a right triangle because of the 90° change in direction. We know the lengths of the legs of the right triangle.



We labeled one of the acute angles θ. The side opposite θ is 3.5 miles and the side adjacent to θ is 2.3 miles. We can use the tangent function to find θ.

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{3.5}{2.3}$$

$$\theta = \tan^{-1}\frac{3.5}{2.3} \approx 56.7°$$

Now the angle formed by the north-south line through the starting point and the segment between the starting and ending points is

$\theta + 31° \approx 56.7° + 31° = 86.7°$.

The bearing of the ending point from the starting point is S 86.7° W.

✎ *Pencil Problem #2* ✎

2. After takeoff, a jet flies 5 miles on a bearing of N 35° E. Then it turns 90° and flies 7 miles on a bearing of S 55° E. At that time, what is the bearing of the jet, to the nearest tenth of a degree, from the point where it took off?

Objective #3: Model simple harmonic motion.

✔ *Solved Problem #3*

3a. A ball on a string is pulled 6 inches below its rest position and then released. The period for the motion is 4 seconds. Write the equation for the ball's simple harmonic motion.

At $t = 0$ seconds, $d = -6$ inches. We use a negative value for d because the motion begins with the ball below its rest position. Also, because the motion begins when the ball is at its greatest distance from rest, we use the model containing cosine rather than the one containing sine.

We need to find values for a and ω in $d = a\cos\omega t$.

Since the maximum displacement is 6 inches and the ball is initially below its rest position, $a = -6$.

The period is given as 4 seconds. Use the period formula to find ω.

$$\frac{2\pi}{\omega} = 4$$

$$2\pi = 4\omega$$

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

The equation is $d = -6\cos\frac{\pi}{2}t$.

✎ *Pencil Problem #3* ✐

3a. A ball on a string is pulled 8 inches below its rest position and then released. The period for the motion is 2 seconds. Write the equation for the ball's simple harmonic motion.

3b. An object moves in simple harmonic motion described by $d = 12\cos\frac{\pi}{4}t,$ where t is measured in seconds and d in centimeters. Find the maximum displacement, the frequency, and the time required for one cycle.

The maximum displacement is the amplitude. Because a = 12, the maximum displacement is 12 centimeters.

The frequency, f, is

$$f = \frac{\omega}{2\pi} = \frac{\frac{\pi}{4}}{2\pi} = \frac{\cancel{\pi}}{4} \cdot \frac{1}{2\cancel{\pi}} = \frac{1}{8}.$$

The frequency is $\frac{1}{8}$ oscillation per second.
The time required for one cycle is the period.

$$\text{period } = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{4}} = 2\cancel{\pi} \cdot \frac{4}{\cancel{\pi}} = 8$$

The time required for one cycle is 8 seconds.

3b. An object moves in simple harmonic motion described by $d = 5\cos\frac{\pi}{2}t,$ where t is measured in seconds and d in inches. Find the maximum displacement, the frequency, and the time required for one cycle.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $B = 66.5°$, $a \approx 4.35$, $c \approx 10.90$ *(4.8 #1)* **1b.** 2059 feet *(4.8 #41)* **1c.** 15.1° *(4.8 #47)*
1d. 33.7 feet *(4.8 #49)*

2. N 89.5° E *(4.8 #57)*

3a. $d = -8\cos \pi t$ *(4.8 #18)*

3b. maximum displacement: 5 inches; frequency: $\frac{1}{4}$ oscillation per second; time for one cycle: 4 seconds *(4.8 #21)*